Q. If we have a "nice" function $f(x)$, how do we construct a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ which converges to $f(x)$ near $x = a$?

Q. First, why do we care?

- Computation: partial sums are easy for calculators/computers to compute

Q. Okay, now answer the question.

① Relate (via algebra or calculus) $f(x)$ to $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

→ ② Something Else

Note: ↓ are exs. of ①

Ex 1 $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $x \in (-1, 1)$

Ex 2 $\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$ for $x \in (-1, 1)$

Ex 3 $\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ for $x \in (0, 2)$

Ex 4 $\arctan(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} x^{2m+1}$ for $x \in (-1, 1)$

Q. What makes it possible find power series for these functions? For which "nice" functions doesn't this approach work?

- Can do the above examples b/c they relate to some rational function ($\frac{\text{poly.}}{\text{poly.}}$)

- Won't work for: e^x, $\sin(x)$, $\cos(x)$

Q. What is a better approach for finding the coefficients of a power series representation of $f(x)$? Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all x in $(a-R, a+R)$.

PQ. What is $f(a)$?

(a) 0 (b) c_0 (c) c_1 (d) a

$f(a) = c_0(a-a)^0 + c_1(a-a) + c_2(a-a)^2 + \cdots$

$= c_0$

$f(x) = f(a) + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

$f'(x) = 0 + c_1 \cdot 1 + c_2 \cdot 2 \cdot (x-a) + 3c_3(x-a)^2 + \cdots$

$f'(a) = c_1 + 0 + 0 + 0 + \cdots$

$f''(x) = 2c_2 + 3 \cdot 2 c_3(x-a) + 4 \cdot 3 c_4(x-a)^2 + \cdots$

$f''(a) = 2c_2 + 0 + 0 + 0 + \cdots$

$\hookrightarrow \frac{f''(a)}{2} = c_2$

$f'''(x) = 3 \cdot 2 c_3 + 4 \cdot 3 \cdot 2 \cdot (x-a) + \cdots$

$\hookrightarrow f'''(a) = 3 \cdot 2 \cdot c_3 \rightarrow \frac{f'''(a)}{3 \cdot 2} = c_3$

$\frac{f^{(n)}(a)}{n!} = c_n$

Def: Suppose f is a function that is Smooth at $x = a$ (i.e. $f(a), f'(a), f''(a), f^{(3)}(a), \ldots$ all exist). The n^{th} Taylor coefficient for f at a is $\dfrac{f^{(n)}(a)}{n!}$ and the Taylor series for f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$$

If $a = 0$ it is sometimes called a Maclaurin Series.

"Thm:" If $f(x)$ is "nice," then the Taylor series for $f(x)$ converges to $f(x)$ on its ~~radius~~ interval of convergence.

"Proof:" Later? For specific functions...

Ex 5 Find the Taylor series for $f(x) = e^x$ centered at 0, and find its interval of convergence.

PQ. Fix an integer $n \geqslant 0$. What is $f^{(n)}(0)$?

(a) 1 (b) $\frac{1}{n!}$ (c) 0 (d) $n!$

$f^{(n)}(x) = e^x \longrightarrow f^{(n)}(0) = e^0 = 1$

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

$\Rightarrow$ Taylor series for e^x centered at 0 is

$$\sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 < 1 \text{ for all } x$$

$\Rightarrow$ Interval of convergence is $(-\infty, \infty)$

(3) Interval of Convergence

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(2n)!}{(2n+2)!} \cdot \frac{x^{2n+2}}{x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| x^2 \cdot \frac{(2n)(2n-1)\cdots 2\cdot 1}{(2n+2)(2n+1)(2n)(2n-1)\cdots 2\cdot 1} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right|$$

$$= 0 < 1$$

$\Rightarrow$ Interval of convergence $= (-\infty, \infty)$

Ex 6 Find the Taylor series for $f(x) = \cos(x)$ centered at 0 and find its interval of convergence.

(1) Find coefficients

$$C_n = \frac{f^{(n)}(0)}{n!}$$

$f(x) = \cos(x)$; $f'(x) = -\sin(x)$; $f''(x) = -\cos(x)$; $f'''(x) = \sin(x)$

$f^{(4)}(x) = \cos(x)$, ... $f^{(n)}(x) =$

$$\begin{cases} \cos(x) & n = 4k \\ -\sin(x) & n = 4k+1 \\ -\cos(x) & n = 4k+2 \\ \sin(x) & n = 4k+3 \end{cases}$$

$$f^{(n)}(0) = \begin{cases} 1 & n = 4k \\ 0 & n = 4k+1 \\ -1 & n = 4k+2 \\ 0 & n = 4k+3 \end{cases}$$

$$c_n = \frac{f^{(n)}(0)}{n!}$$

(2) Create Taylor series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$= \frac{1}{0!} + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \frac{-1}{6!}x^6 + \frac{0}{7!}x^7 + \frac{1}{8!}x^8 + \cdots$$

$$= 1 + \frac{-1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{-1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

"Oiler"

Cool Stuff: Euler's Formula

Recall: $i = \sqrt{-1}$

$\hookrightarrow i^0 = 1$, $i^1 = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = i^3 \cdot i = -i \cdot i = -(-1) = 1$

$i^5 = \sqrt{-1}$, $i^6 = -1$, $i^7 = -i$

$$\hookrightarrow i^n = \begin{cases} 1 & n = 4k \\ i & n = 4k+1 \\ -1 & n = 4k+2 \\ -i & n = 4k+3 \end{cases}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$t \in \mathbb{R}$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \frac{i^0 t^0}{0!} + \frac{i^1 t}{1!} + \frac{i^2 t^2}{2!} + \frac{i^3 t^3}{3!} + \cdots$$

$$= 1 + \frac{it}{1!} + \frac{(-1)t^2}{2!} + \frac{(-i)t^3}{3!} + \frac{(1)t^4}{4!} + \frac{it^5}{5!} + \frac{(-1)t^6}{6!} + \frac{(-i)t^7}{7!} + \cdots$$

$$= 1 + \frac{(-1)t^2}{2!} + \frac{t^4}{4!} + \frac{(-1)t^6}{6!} + \frac{t^8}{8!} + \cdots$$

$$+ \frac{it}{1!} + \frac{(-i)t^3}{3!} + \frac{it^5}{5!} + \frac{-it^7}{7!} + \cdots$$

$$= \cos(t) + i\left(\frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right)$$

$$= \cos(t) + i\sin(t)$$

$$\Rightarrow \boxed{e^{it} = \cos(t) + i\sin(t)}$$

$$\rightarrow e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i \cdot 0 = -1$$

$$\hookrightarrow e^{i\pi} + 1 = 0$$

Q. So why is it that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$?

What does it mean for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$?

A. $e^x = \lim_{k\to\infty} \sum_{n=0}^{k} \frac{x^n}{n!}$

$e^x - \lim_{k\to\infty} \sum_{n=0}^{k} \frac{x^n}{n!} = 0$

$\lim_{k\to\infty} \left(e^x - \sum_{n=0}^{k} \frac{x^n}{n!} \right) = 0$

$R_k(x)$ $P_k(x)$

Def: Let $f(x)$ be a smooth function at $x = a$. The kth partial sum of the Taylor Series for f centered at a, namely

$$\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!} \cdot x^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

2nd T.P.

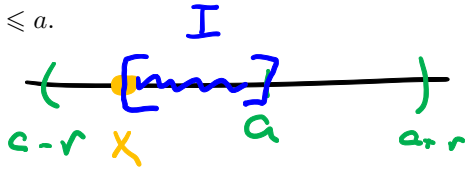
0th T.P.

1st T.P.

is called the k^{th} Taylor polynomial. The k^{th} remainder is the function

$$R_k(x) := f(x) - p_k(x)$$

Thm: (Taylor's Theorem with Remainder) Suppose f is $(k+1)$-times differentiable on an interval $(a-r, a+r)$ with $x \in (a-r, a+r)$. Let I denote the interval $[a, x]$ if $x \geqslant a$ or $[x, a]$ if $x \leqslant a$.



(a) There is a number $c \in I$ such that $R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$

(b) If $|f^{(k+1)}(z)| \leqslant M$ for all $z \in I$, then

$$|R_k(x)| \leq \frac{M}{(k+1)!} \cdot |x-a|^{k+1}$$

$R_k(x)$???

↳ $f(x) = e^x$

↳ $p_k(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^k}{k!}$

$k = 0$

$p_0(x) = 1 \longrightarrow R_0(x) = e^x - 1$

$\left(= \left(1 + x + \frac{x^2}{2} + \cdots\right) - 1 \right)$

$\left(= x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \right)$

$k = 1$

$p_1(x) = 1 + x \longrightarrow R_1(x) = e^x - (1 + x)$

$\left(= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots - (1 + x) \right)$

$\left(= \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \right)$

"$R_k(x) \approx \frac{x^{k+1}}{(k+1)!} + \frac{x^{k+2}}{(k+2)!} + \cdots$ "

Ex 7 Use the Maclaurin series for $f(x) = e^x$ to estimate (i) e^1, (ii) e^5, (iii) e^{15} each with error < 0.0001.

Maclaurin series: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

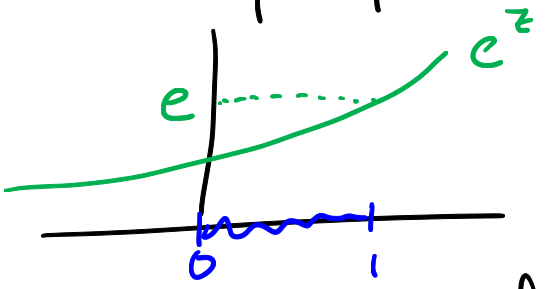
(i) $e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} \rightarrow x = 1, a = 0 \rightarrow$ I = [0, 1]

Want: $|R_k(1)| < 0.0001$

Know: If $|f^{(k+1)}(z)| \leq M$, then $|R_k(1)| \leq \dots$ for $z \in I$

↳ Can I find M?

Want: $|e^z| \leq M$ for $z \in I$



Know: $|e^z| \leq e$ for $z \in I$ (M)

Know: $|R_k(x)| \leq \frac{M}{(k+1)!} \cdot |x-a|^{k+1}$

$|R_k(1)| \leq \frac{e}{(k+1)!} \cdot |1-0|^{k+1} = \frac{e}{(k+1)!}$

Now: If $\frac{e}{(k+1)!} < 0.0001$, then

So is $|R_k(1)|$

Try (on your own): $k = 7$

$$e^1 \approx P_7(1) = \sum_{n=0}^{7} \frac{1}{n!} \approx$$

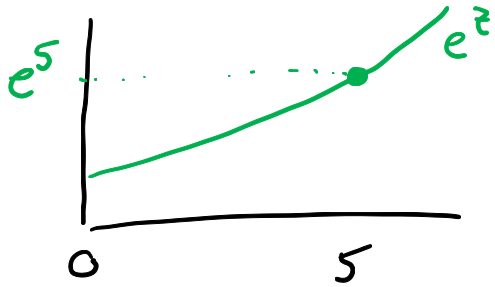
(ii) $e^5 = \sum_{n=0}^{\infty} \frac{5^n}{n!}$

↳ $x = 5$, $a = 0$ ⟶ 

Want: $|R_k(5)| < 0.0001$

Know: If $|f^{(k+1)}(z)| \le M$ for $z \in [0,5]$, then....

↳ Find M? $f^{(k+1)}(z) = e^z$



⟶ $|f^{(k+1)}(z)| \le e^5$ "M" for $z \in [0,5]$

Know: $|R_k(5)| \le \frac{e^5}{(k+1)!} \cdot |5-0|^{k+1} = \frac{e^5 \cdot 5^{k+1}}{(k+1)!}$

Goal: $\frac{e^5 \cdot 5^{k+1}}{(k+1)!} < 0.0001$ Trial-and-error: $k = 21$

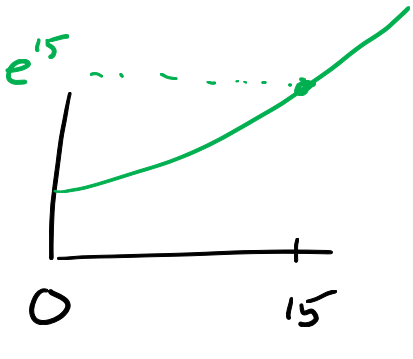
$$e^5 \approx p_{21}(5) = \sum_{n=0}^{21} \frac{5^n}{n!}$$

(iii) $e^{15} \rightarrow x = 15,\ a = 0 \rightarrow I = [0, 15]$

Want: $|R_k(15)| < 0.0001$

Know: If $|f^{(k+1)}(z)| \leq M$, then ....

for $z \in [0, 15]$



$$|f^{(k+1)}(z)| = e^z$$

$$\leq e^{15} \quad \text{for } z \in [0, 15]$$

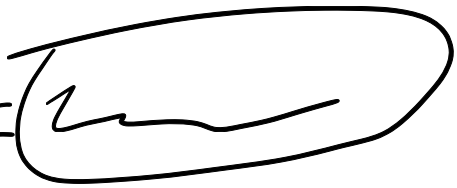
$e^{15} = M$

Know: $|R_k(15)| \leq \frac{e^{15}}{(k+1)!} \cdot |15 - 0|^{k+1} = \frac{e^{15} \cdot 15^{k+1}}{(k+1)!}$

Goal: $\frac{e^{15} \cdot 15^{k+1}}{(k+1)!} < 0.0001$

Trial-and-Error: $k = 56$ works

$$e^{15} \approx p_{56}(15) = \sum_{n=0}^{56} \frac{15^n}{n!} =$$



Summary: (a) If f is smooth at a, then we can compute the Taylor series at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(b) Using the ratio test, we can find a value r such that the Taylor series converges for all x in $(a-r, a+r)$.

(c) Given x in $(a-r, a+r)$, Taylor's Theorem tells us:

If $|f^{(k+1)}(z)| \leq M$ for all z between x and a,
then $|R_k(x)| \leq \frac{M}{(k+1)!} \cdot |x-a|^{k+1}$

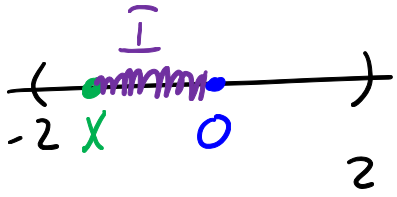
Observe: The greater the distance between x and a, the larger M might need to be and the larger $|x-a|$ is, so the larger k has to be.

Ex 8 Why is the Maclaurin series for $\cos(x)$ equal to $\cos(x)$?

Ex 9 Find a value of k for which the kth Taylor polynomial for e^x is within 0.001 of e^x for every $x \in (-2, 2)$.

centered at $a = 0$ or $[-2, 2]$

Want: $|R_k(x)| < 0.001$ for every $x \in (-2, 2)$



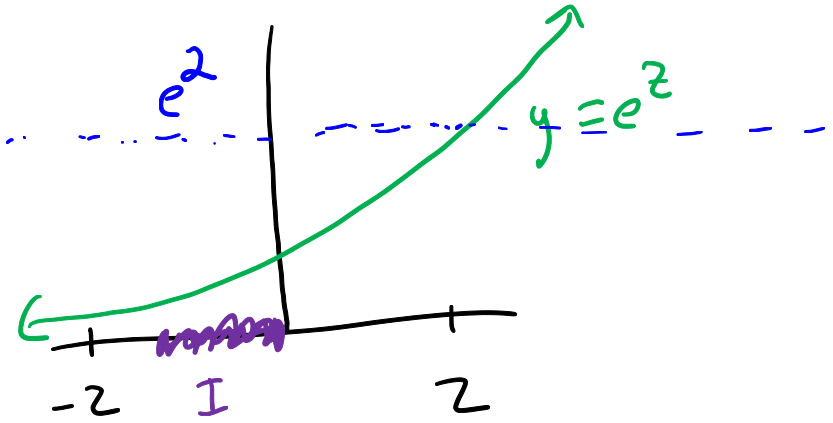
Now: If $|f^{(k+1)}(z)| \le M$, then $|R_k(x)| \le \frac{M}{(k+1)!}|x-a|^{k+1}$ for all $z \in I$

If $|e^z| \le M$, then $|R_k(x)| \le \frac{M}{(k+1)!}|x|^{k+1}$ for all $z \in I$ (M = e^2)

$$|R_k(x)| \le \frac{e^2}{(k+1)!}|x|^{k+1}$$

$$\le \frac{e^2}{(k+1)!} \cdot 2^{k+1} < 0.001$$

$k = 10$ works



$f(x) = \sin(x)$

$$\downarrow$$

$$\left| f^{(k+1)}(x) \right| = \left| \pm \sin(x) \right| \text{ or } \left| \pm \cos(x) \right|$$

